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Unified Theory of Gravity and Electromagnetism: Classical and Quantum Aspects

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ABSTRACT

A unified classical theory of gravity and electromagnetism with a torsion vector $\Gamma_i \neq 0$, proposed by S N Bose in 1952, is introduced. In this theory, the torsion vector acts as a magnetic current and it is shown that (i) the electromagnetism is invariant under continuous Heaviside–Larmor transformations and (ii) the electric and magnetic charges are topologically quantised, satisfying the Dirac quantisation condition, without implying any Dirac string provided Γ_i is curl-less.

Keywords: Toplogical Charges, Bianchi Identity

#1 <u>Introduction:</u> -

Einstein's famous equation, $R_{ik} - \frac{1}{2}g_{ik}R = \kappa T_{ik}$.

$$(1)$$

Of general relativity(GR) has been extremely successful in explaining and predicting various weak-field phenomena such as the precession of the perihelion of *mercury*, the bending of light by stars, the Shapiro delay time and the frame-dragging precession of gyroscopes measured by the gravity probe B experiment [1]. In the strong field sector, the observation of gravitational waves originating in the merger of binary black holes [2] has also come as a reassurance of the general correctness of the theory.

The successful weak-field predictions are solutions of the field equations $R_{ik}=0$, i.e. with $T_{ik}=0$ 'outside' a spherically symmetric body having mass and angular momentum, such as the Schwarzschild and Kerr metrics [3]. Hence, $R_{ik}=0$ are not necessarily 'Vacuum equations' in the sense of a completely empty Universe. On the other hand, attempts to use solutions of equation (1) with $T_{ik}\neq0$ in Friedmann-Lemaitre-Robertson-Walker (FLRW) cosmology have led to many intractable problems such as the hypothetical non-baryonic dark matter [4], dark

energy and the cosmological constant problem [5], the horizon problem [6] and the flatness problem [7] which show no signs of going away.

Einstein himself was very unhappy with the role that the stress-energy tensor T_{ik} played in GR. He had repeatedly emphasised that it was only phenomenological representation of matter, to be regarded with caution. In 1936 he wrote [8]: "[General Relativity] is sufficient –as far as we know for the representation of the observed facts of celestial mechanics. But it is similar to a building, one wing of which is made of fine marble (left part of the equation), but the other wing of which is built of low grade wood (right side of the equation). The phenomenological representation of matter is in fact, only a crude substitute for a representation which would do justice to all known properties of matter".

Also, in a letter to Michele Besso [9] he wrote: "But it is questionable whether the equation $R_{ik} - \frac{1}{2} g_{ik} R = \kappa T_{ik}$ has any reality left within it in the face of quanta. I vigorously doubt it. In contrast, the left hand side of the equation surely contains a deeper truth. If the equation $R_{ik}=0$ really determines the behaviour of the singularities, then a law describing this behaviour would be justified far more deeply than the aforementioned equation, which is not unified and only phenomenologically justified".

This quotes show that the stress-energy tensor was unsatisfactory to Einstein for two reasons. First, it is not geometrical in nature like the left side of equation (1) and hence not unified with it, and secondly, it does not reflect the quantum nature of matter and radiation. It is merely phenomenological, a place holder for a more satisfactory theory of matter. This is why later on Einstein preferred to work with the equation $R_{ik}=0$ in which the matter appears as singularities and follows geodesics, through even this was a placeholder for a more satisfactory future theory of matter[10,11].

It was therefore natural for him to try and construct a unified geometrical theory of all fields with the hope that the quantum features would emerge as consequences of the nonlinearity of the theory. The only known long-range interactions being electromagnetic and gravity, he sought to bring them under one umbrella. Now, in GR the number of independent variables is 10 (the 10 components of the metric tensor $g_{\mu\nu}$). Hence, in order to incorporate electromagnetism into a unified theory, one needed additional variables. There were many options for this. After Weyl's and Kaluza's attempts at unification, it was Eddington[12] who first proposed to replace the metric as a fundamental concept by a non-symmetric affine connection Γwhich could then be split into a symmetric and an antisymmetric parts. However, Einstein [13], supported by Schrödinger [14], extended the idea to include also a non-symmetric g. Just as passing beyond the Euclidean geometry gravitation makes its appearance, so going

beyond the Riemannian geometry electromagnetism appears 'naturally' as the antisymmetric part of the metric 'without requiring any higher dimensional space'.

A problem with such unified theories is that the symmetry between gravity and electromagnetism is actually badly broken in the universe, the electromagnetic interaction being enormously stronger than gravity ($\frac{\alpha}{\alpha_G} = 10^{36}$)[15]. In high-energy physics, where unified gauge field theories have proved extremely successful, the differences in coupling strengths and masses emerge as effects of spontaneous symmetry breaking, i.e. symmetries broken by the ground state of the system, not the Lagrangian. In Classical unified theories of gravity and electromagnetism with a stress tensor T_{ik} representing matter/energy, the problem can be easily taken care of by simply multiplying the electromagnetic source current by a dimensional parameter while retaining the full geometric symmetry of the Lagrangian and therefore the unity of the two forces. However, in theories without the matter/energy stress tensor preferred by Einstein, this is not possible, and some geometric symmetry has to be broken to generate the stress tensor with a source current as a purely geometric aspect of the nonlinear field equations. The purpose of this paper is to consider one such theory (to be made more explicit in **#3**) and work out some of its striking consequences. To lead up to it, a unified theory with maximal symmetry will be presented first in the next section to set the scene for the non-maximal variant.

#2 A maximally symmetric unified theory with T_{ik}=0

Let U_4 be a smooth manifold with signature (-, +, +, +) and endowed with a non-symmetric affine connection Γ and a non-symmetric metric g. Let

$$R_{ik} = \Gamma^{\alpha}_{ik,\alpha} - \Gamma^{\alpha}_{i\alpha,k} + \Gamma^{\xi}_{ik}\Gamma^{\lambda}_{\xi\lambda} - \Gamma^{\xi}_{i\lambda}\Gamma^{\lambda}_{\xi k} \qquad (2)$$

be the non-symmetric curvature tensor. Let us also define

$$\Gamma^{\lambda}_{(ik)} = \frac{1}{2}(\Gamma^{\lambda}_{ik} + \Gamma^{\lambda}_{ki}), \qquad (3)$$

$$\Gamma^{\lambda}_{[ik]} = \frac{1}{2}(\Gamma^{\lambda}_{ik} - \Gamma^{\lambda}_{ki}), \qquad (4)$$

$$\Gamma_{l} = \frac{1}{2}(\Gamma^{\lambda}_{l\lambda} - \Gamma^{\lambda}_{\lambda l}), \qquad (5)$$

$\Gamma^{\lambda}_{[ik]}$ is called the Cartan torsion tensor. Following Einstein [16], let us put $\Gamma_l = 0$ because it is not determined by any equation in the theory. Similarly, let

$$\bar{g}^{(ik)} = \frac{1}{2}\sqrt{-g}(g^{ik} + g^{ki}), \qquad (6)$$

$$\overline{g}^{[ik]} = \frac{1}{2}\sqrt{-g}(g^{ik} - g^{ki}), \tag{7}$$

To restrict the number of possible covariant terms in a non-symmetric theory, Einstein [16] imposed 'transposition invariance' and 'λ-transposition invariance' on the theory. Let $\tilde{\Gamma}^{\lambda}_{ik} = \Gamma^{\lambda}_{ki}$ and $\tilde{g}_{ik} = g_{ki}$. Then terms those are invariant under the simultaneous replacements of Γ^{λ}_{ik} and g_{ik} by $\tilde{\Gamma}^{\lambda}_{ik}$ and $\tilde{g}_{ik}$,respectively are called transposition invariants. For example, the tensor $R_{ik}(2)$ is not transposition invariant because it is transposed to

$$\tilde{R}_{ik} = \Gamma^{\alpha}_{ki,\alpha} - \Gamma^{\alpha}_{\alpha i,k} + \Gamma^{\xi}_{ki}\Gamma^{\lambda}_{\xi\lambda} - \Gamma^{\xi}_{\lambda i}\Gamma^{\lambda}_{k\xi} \ . \tag{8}$$

Next, define the transformations

$$\Gamma^{i\prime}_{kl} = \Gamma^{i}_{kl} + \delta^{i}_{k}\lambda, l,$$

$$g^{ik\prime} = g^{ik}, \tag{9}$$

Where λ is an arbitrary function of the co-ordinates. Then R_{ik}(eq. (2)) is λ-transformation invariant (or projective invariant). What this means is that a characterised by R_{ik} cannot determine the Γ-field completely but only up to an arbitrary function λ. Hence, in such a theory, Γ and Γ' represent the same field. Further, this 'λ-transposition' produces a non-symmetric Γ' from a Γ that is symmetric or anti-symmetric in the lower indices. Hence, the symmetry condition for Γ loses its objective significance. This sets the ground for a genuine unification of gravity and electromagnetism, the former determined by the symmetric par and the latter by the anti-symmetric part of the action.

Let us write the simplest transposition invariant and λ-transposition invariant Lagrangian [17]:

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}(g^{ik}R_{ik} + \tilde{g}^{ik}\tilde{R}_{ik}), \tag{10}$$

Which can be expressed as (see Appendix 1)

$$\mathcal{L} = \overline{g}^{ik}(R_{(ik)} - \Gamma^{\lambda}_{[i\xi]}\Gamma^{\xi}_{[\lambda k]}) + \overline{g}^{[ik]}\Gamma^{\lambda}_{[ik];\lambda} \tag{11}$$

$$:= \overline{g}^{(ik)}R'_{(ik)} + \overline{g}^{[ik]}\Gamma^{\lambda}_{[ik];\lambda} \tag{12}$$

The variation of the action $\int \mathcal{L}d^4x$ holding $\overline{g}^{ik}$ and $\overline{g}^{[ik]}$ constant at once implies

$$R'_{(ik)} = R_{(ik)} - \Gamma^{\lambda}_{[i\xi]}\Gamma^{\xi}_{[\lambda k]} = 0, \tag{13}$$

$$\Gamma^{\lambda}_{[ik];\lambda} = 0. \tag{14}$$

It can also be shown (see Appendix 1) using the variational principle that the equation g and Γ in this non-symmetric theory is,

$$\overline{g}^{ik}_{,\lambda} + \overline{g}^{i\alpha}\Gamma'^k_{\lambda\alpha} + \overline{g}^{\alpha k}\Gamma'^i_{\alpha\lambda} - \overline{g}^{ik}\Gamma^\alpha_{(\lambda\alpha)} = 0, \tag{15}$$

Where,

$$\Gamma'^k_{\lambda\alpha} = \Gamma^k_{(\lambda\alpha)} + \Gamma^k_{[\lambda\alpha]},$$

$$\Gamma'^i_{\alpha\lambda} = \Gamma^i_{(\alpha\lambda)} + \Gamma^i_{[\alpha\lambda]}$$

and further that $\overline{g}^{[i\alpha]}_{,\alpha} = 0$. $\qquad\qquad$ (16)

The last equation can be interpreted as Maxwell's equations for electrodynamics by identifying $\overline{g}^{[ik]}$ with the dual electromagnetic field $\widetilde{F}^{ik}$. The electric current is given by,

$$j^i = \frac{1}{3!}\epsilon^{i\nu\lambda\rho}(\overline{g}_{[\nu\lambda],\rho} + \overline{g}_{[\lambda\rho],\nu} + \overline{g}_{[\rho\nu],\lambda})$$

$$= 0. \tag{17}$$

The current vanishes because of the Bianchi identity in the first line. Hence, this theory describes the free electromagnetic fields with the same strength as gravity. The equation sets (13)-(16) are the fundamental equations of the theory. Note that the Ricci tensor $R'_{(ik)}$ in the theory, which is flat, has an additional term compared to the GR Ricci tensor $R_{(ik)}$. Clearly, the additional curvature has its origin in the torsion in the manifold U_4.

In this context, the scepticism expressed by Pauli concerning a unified theory of this kind in his classic book 'Theory of relativity' [18] is worth recalling: "Whether the field equations of this theory, which are based on the formal postulates of λ-invariance and of transposition invariance without any obvious geometrical and physical meaning, can actually be connected with physics at all, is rather doubtful". We shall see in the next section how the connection to physics can be established by breaking one of the symmetries, namely λ-invariance.

#3 A non-maximally unified theory with $T_{ik}=0$, $\Gamma_i \neq 0$

In this section we shall relax the Einstein condition $\Gamma_i=0$ and consider a more general form of unified theory proposed by Bose [17] in 1952. Bose's Lagrangian,

$$\mathcal{L} = \sqrt{-g}\,[g^{ik}R_{ik} + \widetilde{g}^{ik}\widetilde{R}_{ik}] + ag^{-(ik)}\Gamma_i\Gamma_k + b\,\widetilde{g}^{[ik]}(\Gamma_{i,k} - \Gamma_{k,i}) \tag{18}$$

has two additional terms to the maximally symmetric Lagrangian (10) with *a* and *b* being two arbitrary dimensional parameters. Projective or λ-invariance would require *a* to vanish but not

b. Since, $a \neq 0$ in Bose's theory, it is only transposition invariant. Equations (13) and (14) are modified to

$$\mathcal{R}_{ik} = R_{(ik)} - Q_{i\xi}^\lambda Q_{\lambda k}^\xi + x \Gamma_i \Gamma_k$$

$$= R_{(ik)} - \Gamma_{[i\xi]}^\lambda \Gamma_{[\lambda k]}^\xi + a \Gamma_i \Gamma_k = 0, \tag{19}$$

$$Q_{[ik];\lambda}^\lambda - y\,(\Gamma_{i,k} - \Gamma_{k,i}) = 0, \tag{20}$$

Where,

$$Q_{ik}^\lambda = \Gamma_{[ik]}^\lambda + \frac{1}{3}\delta_i^\lambda \Gamma_k - \frac{1}{3}\delta_k^\lambda \Gamma_i, \tag{21}$$

$$Q_{ik;\lambda}^\lambda = Q_{ik,\lambda}^\lambda - Q_{i\xi}^\lambda \Gamma_{(\lambda k)}^\xi - Q_{\xi k}^\lambda \Gamma_{(i\lambda)}^\xi + Q_{ik}^\xi \Gamma_{(\xi k)}^\lambda \tag{22}$$

and $x = a + (\frac{1}{3})$, $y = (\frac{1}{6}) - b$. Hence, the new Ricci tensor $\mathcal{R}_{ik}$ is flat though the universe has other fields than gravity, and is hence not empty. Note that one can also write equation (19) in the form

$$R_{(ik)} = (\Gamma_{[i\xi]}^\lambda \Gamma_{[\lambda k]}^\xi - a \Gamma_i \Gamma_k) := \kappa\,(T_{ik} - \frac{1}{2} g_{ik} T), \tag{23}$$

where $\kappa = \frac{8\pi G}{c^4}$ is the Einstein constant, which shows that $\kappa^{-1}(\Gamma_{[i\xi]}^\lambda \Gamma_{[\lambda k]}^\xi - a \Gamma_i \Gamma_k)$ can be interpreted as a traceless stress-tensor. As will be seen presently, the torsion vector Γ_i is basically the source current of the dual electromagnetic field.

Equation (19) will lead to corrections to the Schwarzschild and Kerr metrics that are analogous to the Reissner-Nordstrom and Kerr-Newman metrics.

The variational principle is a little more complex because $Q_{i\lambda}^\lambda = 0$, and all the 24 components of Q_{ik}^λ are not independent, and consequently one has to use asset of Lagrange multipliers k^i [17]. As shown in Appendix B the equations connecting the g's and Γ's in this case are of the form

$$g_{,\lambda}^{ik} + g^{i\alpha} \Gamma_{\lambda\alpha}^{"k} + g^{\alpha k} \Gamma_{\alpha\lambda}^{"i} = 3 g^{ik} \Phi_\lambda, \tag{24}$$

$$\Phi_\lambda = \frac{1}{2} g_{[\lambda\beta]} k^\beta, \tag{25}$$

where,

$$\Gamma_{}^{"k} = \Gamma_{(\lambda\alpha)}^k + Q_{\lambda\alpha}^k + \frac{1}{\sqrt{-g}}\,(g_{\lambda\beta} k^\beta \delta_\alpha^k - g_{\beta\alpha} k^\beta \delta_\lambda^k),$$

$$k^\beta = -\frac{1}{2}\left(\frac{x}{3y}\right)\sqrt{-g}\,g^{(\beta\alpha)} \Gamma_\alpha. \tag{26}$$

Equation (16) is modified to (see Appendix B)

$$\bar{g}^{[i\alpha]}_{,\alpha} = 6k^i = 3\theta\ \bar{g}^{(ij)}\ \Gamma_j\ , \qquad \theta = -\left(\frac{x}{3y}\right), y \neq 0. \tag{27}$$

Thus, Γ^i turns out to be the source of dual electromagnetic field unless $x=0$, i.e. $a=-\frac{1}{3}$ and $k^i=0$. If $\Gamma_l=k_l=0$, one gets back the maximally unified theory. By multiplying Γ_i by the dimensional parameter $\zeta=\sqrt{\mu_0/k}$, where μ_0 is free space permeability, we get,

$$\tilde{F}^{i\alpha}_{,\alpha}=j^i_m, \tag{28}$$

Where $j^i_m=3\theta\zeta\Gamma^i$ is the magnetic source current which is automatically conserved because $\tilde{F}^{i\alpha}=-\tilde{F}^{\alpha i}$. Defining $F_{kl}=\epsilon_{kli\alpha}\tilde{F}^{i\alpha}$, we have

$$\partial^k F_{kl}=\partial^k \epsilon_{kli\alpha}\tilde{F}^{i\alpha}:=j_l, \tag{29}$$

Where j_l is the electric source current which is also automatically conserved. Hence, using the definitions $\alpha_G = Gm_p^2/\hbar c$ and $\alpha=e^2/\hbar c$, we get $\frac{\alpha}{\alpha_G} = \frac{e^2}{m_p^2} = 10^{36}$ in Planck units in which $G=c=\hbar=4\pi\epsilon_0=1$.

Equations (28) and (29) constitute the complete set of Maxwell equations in the presence of the two geometric source currents:

$$\vec{\nabla}\times\vec{B}-\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}=\vec{j}, \qquad \vec{\nabla}.\vec{E}=\rho_e, \tag{30}$$

$$\vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=-\vec{j}_m, \qquad \vec{\nabla}.\vec{B}=\rho_m. \tag{31}$$

The two Bianchi identities that must be satisfied are

$$F_{\mu\nu,\lambda} + F_{\nu\lambda,\mu} + F_{\lambda\mu,\nu} = 0, \tag{32}$$

$$\tilde{F}_{\mu\nu,\lambda} + \tilde{F}_{\nu\lambda,\mu} + \tilde{F}_{\lambda\mu,\nu} = 0, \tag{33}$$

These identities are consistent with the inhomogeneous Maxwell equations provided one makes the following identifications:

$$F^{0i}=-(E^i-E'^i), \qquad F^{ij}=-\epsilon^{ijk}(B_k-B'_K), \tag{34}$$

$$\tilde{F}^{0i}= (B^i-B'^i), \qquad \tilde{F}^{ij}=-\epsilon^{ijk}(E_k-E'_K), \tag{35}$$

Where $(\vec{E}', \vec{B}')$ are auxiliary fields which also satisfy the inhomogeneous Maxwell equations:

$$\vec{\nabla}.\vec{E}'=\rho_e, \qquad \vec{\nabla}\times\vec{E}'+\frac{\partial\vec{B'}}{\partial t}=-\vec{j}_m, \tag{36}$$

$$\vec{\nabla}.\vec{B}'=\rho_m, \qquad \vec{\nabla}\times\vec{B}'-\frac{1}{c^2}\frac{\partial\vec{E}'}{\partial t}=\vec{j}, \tag{37}$$

So that F^{0i} and $\tilde{F}^{0i}$ satisfy the free Maxwell equations. One can define potentials $A^\mu=(\phi/c,\vec{A}),\vec{B}$ and $A'^\mu=(\phi'/c,\vec{A'}),\vec{B}'_m$ through the relations

$$\vec{E}=-\frac{\partial\vec{A}}{\partial t}-\vec{\nabla}\phi, \tag{38}$$

$$\vec{E}'=-\frac{\partial\vec{A'}}{\partial t}-\vec{\nabla}\phi', \tag{39}$$

$$\vec{B}=\vec{\nabla}\times\vec{A}+\vec{B}_m, \qquad\qquad \vec{B}_m=-\vec{\nabla}\tilde{\phi}, \tag{40}$$

$$\vec{B}'=\vec{\nabla}\times\vec{A}'+\vec{B}'_m, \qquad\qquad \vec{B}'_m=-\vec{\nabla}\tilde{\phi}', \tag{41}$$

With the conditions

$$\Box\,\vec{A}=\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=-\vec{j},$$

$$\partial_\mu A^\mu=\vec{\nabla}.\,\vec{A}+\frac{1}{c^2}\frac{\partial\phi}{\partial t}=0, \tag{42}$$

$$\Box\,\vec{A}'=\nabla^2\vec{A}'-\frac{1}{c^2}\frac{\partial^2\vec{A'}}{\partial t^2}=-\vec{j},$$

$$\partial_\mu A'^\mu=\vec{\nabla}.\,\vec{A}'+\frac{1}{c^2}\frac{\partial\phi'}{\partial t}=0, \tag{43}$$

$$\nabla^2\phi-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=-\rho_e,$$

$$\nabla^2\phi'-\frac{1}{c^2}\frac{\partial^2\phi'}{\partial t^2}=-\rho_e, \tag{44}$$

$$\frac{\partial\vec{B}}{\partial t}=\frac{\partial\vec{B}'_m}{\partial t}=-\vec{j}_m, \tag{45}$$

So that both sets of Maxwell equations (30), (31), (36) & (37) are satisfied.

Thus, $\vec{A}'=\vec{A}+\vec{K}$ and $\phi'=\phi+k$ where $\vec{K}$ is a constant vector and k a constant scalar. The fields are invariant under the gauge transformation, $\vec{A}\to\vec{A}+\vec{\nabla}\chi,\phi\to\phi-\frac{\partial\chi}{\partial t}$, where χ is any twice differentiable function of $(\vec{x},\,t)$ that satisfies the equation $\Box\,\chi=0$.

Notice that equation (31) implies that the magnetic field $\vec{B}$ can no longer be just $\vec{\nabla}\times\vec{A}$ as in standard electrodynamics - there must be an additional curl free field $\vec{B}_m$ as in (40) satisfying condition (45). This is in accordance with the Helmholtz decomposition theorem which states that any sufficiently smooth vector field falling off to zero at infinity can be uniquely written as

the sum of two vector fields, one divergence – free (solinoidal) and the other curl - free (irrotational). Consequently, the curl – free field $\vec{B}_m$ can be ignored only at the cost of introducing a singularity, as shown by Dirac [19]. Dirac's famous solution is,

$$\vec{A}=\frac{q_m(1-\cos\theta)}{r\sin\theta}\,\hat{\phi},\ \theta=\pi \tag{46}$$

In spherical polar coordinates. This has a line of singularity along the negative Z – axis characterized by $\theta=\pi$. This is the Dirac string. Such a singular solution becomes unnecessary in the presence of the curl-free field $\vec{B}_m$. Using equation (45) one can write

$$\vec{B}_m= -\int_{\vec{x}=\text{const}} \vec{j}_m dt+\vec{B}_0, \qquad\qquad \vec{B}_0 = \text{Constant vector}, \tag{47}$$

So that on using the continuity equation for $\vec{j}_m$ one has

$$\vec{\nabla}.\vec{B}= \vec{\nabla}.\vec{B}'=\rho_m. \tag{48}$$

This solution is different from the ones proposed by Wu and Yang [20] based on fibre bundle theory and by Cabibbo and Ferrari [21] based on a double potential. It is interesting to note that in all these solutions a single potential describing the magnetic field of a monopole throughout all space (as used by Dirac) is replaced by two potentials.

The presence of Γ_i makes the electrodynamics invariant under continuous transformations:

$$\vec{E} \rightarrow \vec{E}'=\vec{E}\cos\theta-\vec{B}\sin\theta, \tag{49}$$

$$\vec{B} \rightarrow \vec{B}'=\vec{E}\sin\theta+\vec{B}\cos\theta, \tag{50}$$

Where $0\leq \theta \leq \frac{\pi}{2}$. Hence,

$$\vec{j}'=\vec{j}\cos\theta-\vec{j}_m\sin\theta, \tag{51}$$

$$\vec{j}'_m=\vec{j} \sin\theta+\vec{j}_m\cos\theta, \tag{52}$$

$$\rho'_e=\rho_e\cos\theta-\rho_m\sin\theta, \tag{53}$$

$$\rho'_m=\rho_e\sin\theta+\rho_m \cos\theta. \tag{54}$$

For $\theta = \frac{\pi}{2}$ one has $\vec{E} \rightarrow -\vec{B}, \vec{B} \rightarrow \vec{E}, (\rho_e,\vec{j})\rightarrow(-\rho_m,-\vec{j}_m), (\rho_m,\vec{j}_m)\rightarrow(\rho_e,\vec{j})$ [22,23]. This shows that there is complete equivalence and continuous freedom in the choice of electric and magnetic quantities.

#3.1Quantisation:-

Note that (ρ_m, ρ_e) are time components of the corresponding four currents which are determined in terms of continuous fields (see equations (17) & (27)). Also, note that no condition has been imposed so far on Γ_i. Instead of imposing the Einstein condition Γ_i =0 if one imposes the weaker condition

$$\Gamma_{i,k} - \Gamma_{k,i}=0. \tag{55}$$

But b≠0, then like $\mathcal{R}_{ik}$ = 0, $Q^\lambda_{[ik];\lambda}$=0 and one immediately gets a very interesting result, namely thatΓ_i, and hence the magnetic source current j$_{mi}$= $\zeta\Gamma_i$ is an 'irrotational' or, curl-less axial vector. Let, S= $\mathbb{R}^3 \setminus \{(0, 0, z \leq 0)| z \in \mathbb{R}\}$ be the usual 3-dimensional space with the negative Z-axis, along which $\vec{\Gamma} \neq 0$removed. Then the curl-less vector $\vec{\Gamma}$=-$\vec{\nabla}\widetilde{\Phi}$, $\nabla^2\Phi$=0 has vortex solutions $\vec{\Gamma}$=$\vec{e}_\phi/r$ where $\vec{e}_\phi$ is a unit vector and the integral over a unit counter – clockwise circular path C in the xy plane enclosing the origin is

$$\frac{1}{2\pi} \oint_c \vec{\Gamma} \cdot \vec{e}_\phi \mathrm{d}\phi=\text{n}, \qquad\qquad \text{n} \in \mathbb{Z}, \tag{56}$$

where n is a winding number which can be interpreted as the number of magnetic charges enclosed by the unit circle. For n=1, one has a single magnetic charge, i.e., a magnetic monopole carrying some charge g. It is a straightforward consequence of Larmor – Heaviside symmetry, there is also an electric monopole, i.e., a particle carrying electric charge *e*. Thus, particles emerge as topological charges. The product *e*g has the dimension of action and the fundamental unit of action in nature being the Planck constant *h*, it follows the *e*g/*h* = constant, which is essentially topological basis of Dirac quantisation.

Concluding Remarks:-

A remarkable feature of Bose's unified theory with T_{ik}=0,$\Gamma_i \neq 0$ is the occurrence of topological charges satisfying the Dirac quantisation condition without implying any Dirac string, provided $\Gamma_i$is curl-less. Thus, Einstein's dream of deriving quanta from a unified geometry theory of continuous fields is at least partially realized. Magnetic monopoles are also predicted by other unified gauge theories [24, 25], and hence the failure so far to detect them remains a deep puzzle in physics. It must be emphasized that if a=b=0 as in theory with maximal symmetry, none of the above results can be derived. Bose proposed his theory in 1952. He worked out many of the mathematical results given in this paper (equation (18) – (27) & Appendix 2). I have endeavored to derive some new and interesting consequences of his theory, namely the two Bianchi identities, the presence of magnetic charges and currents without Dirac strings, the complete equivalence and freedom in the choice of electric and magnetic quantities in the theory, gauge invariance and, most strikingly, the topological

quantisation of charge as a consequence of adding condition (55) on Γ_i not considered by Bose. This type of quantisation is not possible in a theory with $\Gamma_i=0$. The presence of the magnetic current $\zeta\theta\Gamma_i$ also provides a natural classical theory of magnetism in the universe originating in a non-zero torsion vector. Magnetism is ubiquitous in the universe, and primordial magnetic fields are especially important for probing the Physics of the early universe [26].

Appendix 1:-

The straight forward algebra gives

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}(g^{ik}R_{ik}+\tilde{g}^{ik}\widetilde{R}_{ik})$$

$$= [\bar{g}^{(ik)}(R_{ik}+\widetilde{R}_{ki})+ \bar{g}^{[ik]}(R_{ik}-\widetilde{R}_{ki})]$$

$$=\bar{g}^{(ik)}[R_{ik}-\Gamma^{\lambda}_{[i\xi]}\Gamma^{\xi}_{[\lambda k]}]+ \bar{g}^{[ik]}(\Gamma^{\lambda}_{[ik],\lambda}-\Gamma^{\lambda}_{[i\lambda],k}+\Gamma^{\xi}_{ik}\Gamma^{\lambda}_{\xi\lambda}-\Gamma^{\xi}_{i\lambda}\Gamma^{\lambda}_{\xi k}-\Gamma^{\xi}_{ki}\Gamma^{\lambda}_{\lambda\xi}+\Gamma^{\xi}_{\lambda i}\Gamma^{\lambda}_{k\xi})$$

$$= \bar{g}^{(ik)}[R_{ik}-\Gamma^{\lambda}_{[i\xi]}\Gamma^{\xi}_{[\lambda k]}]+ \bar{g}^{[ik]}[\Gamma^{\lambda}_{ik;\lambda}]. \qquad (1A)$$

Following ref. [17] but using $\Gamma_\mu=0$, let

$$\mathcal{L} =H +\frac{dX^{\lambda}}{dx^{\lambda}} \qquad (1B)$$

With

$$X^{\lambda}=\bar{g}^{(ik)}\Gamma^{\lambda}_{(ik)}-\bar{g}^{(i\lambda)}\Gamma^{k}_{(ik)}+\bar{g}^{[ik]}\Gamma^{\lambda}_{[ik]},$$

$$H = -\bar{g}^{(ik)}_{,\lambda}\Gamma^{\lambda}_{(ik)}+\bar{g}^{(i\lambda)}_{,\lambda}\Gamma^{k}_{(ik)}+ \bar{g}^{(ik)}(\Gamma^{\xi}_{(ik)}\Gamma^{\lambda}_{(\xi\lambda)}-\Gamma^{\xi}_{(i\lambda)}\Gamma^{\lambda}_{(\xi k)}-\Gamma^{\lambda}_{[i\xi]}\Gamma^{\xi}_{[\lambda k]})- \bar{g}^{[ik]}_{,\lambda}\Gamma^{\lambda}_{[ik]}+ \bar{g}^{[ik]}[-\Gamma^{\lambda}_{[i\xi]}\Gamma^{\xi}_{(\lambda k)}-\Gamma^{\lambda}_{[\xi k]}\Gamma^{\xi}_{(i\lambda)}+\Gamma^{\xi}_{[ik]}\Gamma^{\lambda}_{(\xi\lambda)}].$$

Thus, *H* is free of the partial derivatives of $\Gamma^{\lambda}_{(ik)}$ and $\Gamma^{\lambda}_{[ik]}$, and the four-divergence term in the action integral is equal to a surface integral at infinity on which all arbitrary variations are taken to vanish.

The variations of *H* with respect to $\Gamma^{\lambda}_{(ik)}$ and $\Gamma^{\lambda}_{[ik]}$ give,

$$\bar{g}^{(ik)}_{,\lambda}+\bar{g}^{(i\alpha)}\Gamma^{k}_{(\lambda\alpha)}+\bar{g}^{(\alpha k)}\Gamma^{i}_{(\alpha\lambda)}-\bar{g}^{(ik)}\Gamma^{\alpha}_{(\lambda\alpha)}= - [\bar{g}^{[i\alpha]}\Gamma^{k}_{[\lambda\alpha]}+\bar{g}^{[\alpha k]}\Gamma^{i}_{[\alpha\lambda]}], \qquad (1C)$$

$$\bar{g}^{[ik]}_{,\lambda}+\bar{g}^{[i\alpha]}\Gamma^{k}_{(\lambda\alpha)}+\bar{g}^{[\alpha k]}\Gamma^{i}_{(\alpha\lambda)}-\bar{g}^{[ik]}\Gamma^{\alpha}_{(\lambda\alpha)}= - [\bar{g}^{(i\alpha)}\Gamma^{k}_{[\lambda\alpha]}+\bar{g}^{(\alpha k)}\Gamma^{i}_{[\alpha\lambda]}], \qquad (1D)$$

Adding (1C) and (1D), we get,

$$\bar{g}^{ik}_{,\lambda} + \bar{g}^{(i\alpha)}(\Gamma^k_{(\lambda\alpha)} + \Gamma^k_{[\lambda\alpha]}) + \bar{g}^{\alpha k}(\Gamma^i_{(\alpha\lambda)}+\Gamma^i_{[\alpha\lambda]})- \bar{g}^{ik}\Gamma^\alpha_{(\lambda\alpha)}=0 \qquad (1E)$$

This can be written as,

$$\bar{g}^{ik}_{,\lambda}+\bar{g}^{i\alpha}\Gamma'^k_{\lambda\alpha}+\bar{g}^{\alpha k}\Gamma'^i_{\alpha\lambda}-\bar{g}^{ik}\Gamma^\alpha_{(\lambda\alpha)}=0,$$

$$\Gamma'^k_{\lambda\alpha}=\Gamma^k_{(\lambda\alpha)}+\Gamma^k_{[\lambda\alpha]}, \qquad\qquad\qquad \Gamma'^i_{\alpha\lambda}=\Gamma^i_{(\alpha\lambda)}+\Gamma^i_{[\alpha\lambda]}. \qquad (1F)$$

This is equation (15).

By contracting (1F) once with respect to (k, λ), then with respect to (i, λ), and subtracting equations term by term, one gets equation (16).

Appendix 2:-

In this appendix we shall use the Greek symbols μ,ν instead of i,k. Define

$$s^{\mu\nu}=\frac{1}{2}\sqrt{-g}(g^{\mu\nu}+ g^{\nu\mu})\equiv \frac{1}{2}(\bar{g}^{\mu\nu}+ \bar{g}^{\nu\mu}).$$

$$=\frac{1}{2}\sqrt{-g}g^{(\mu\nu)}, \qquad (2a)$$

$$a^{\mu\nu}=\frac{1}{2}\sqrt{-g}(g^{\mu\nu}- g^{\nu\mu})\equiv \frac{1}{2}(\bar{g}^{\mu\nu}- \bar{g}^{\nu\mu}).$$

$$=\frac{1}{2}\sqrt{-g}g^{[\mu\nu]}, \qquad (2b)$$

The equations of connection in the broken symmetric theory are obtained by writing

$$\mathcal{L} =H +\frac{dX^\lambda}{dx^\lambda} \qquad (2c)$$

With

$$X^\lambda=s^{\mu\nu}\Gamma^\lambda_{(\mu\nu)}-s^{\mu\lambda}\Gamma^\nu_{(\mu\nu)}+a^{\mu\nu}Q^\lambda_{\mu\nu}+\frac{2}{3}a^{\mu\lambda}\Gamma_\mu+\Gamma^\lambda,$$

$$H =-s^{\mu\nu}_{,\lambda}\Gamma^\lambda_{(\mu\nu)}+s^{\mu\lambda}_{,\lambda}\Gamma^\nu_{(\mu\nu)}+ s^{\mu\nu}(\Gamma^\xi_{(\mu\nu)}\Gamma^\lambda_{(\xi\lambda)}-\Gamma^\xi_{(\mu\lambda)}\Gamma^\lambda_{(\xi\nu)}) + s^{\mu\nu}(-Q^\lambda_{\mu\xi}Q^\xi_{\lambda\nu}+x\Gamma_\mu\Gamma_\nu) +s^{\mu\nu}\Gamma^\lambda_{(\mu\nu)}\Gamma_\lambda- a^{\mu\nu}_{,\lambda}Q^\lambda_{\mu\nu}+ a^{\mu\nu}(-Q^\lambda_{\mu\xi}\Gamma^\xi_{(\lambda\nu)}-Q^\lambda_{\xi\nu}\Gamma^\xi_{(\mu\lambda)}+Q^\xi_{\mu\nu}\Gamma^\lambda_{(\xi\lambda)})- ya^{\mu\lambda}_{,\lambda}\Gamma_\mu,$$

$$Q^\lambda_{\mu\nu} = \Gamma^\lambda_{[\mu\nu]} + \frac{1}{3}\delta^\lambda_\mu \Gamma_\nu - \frac{1}{3}\delta^\lambda_\nu \Gamma_\mu. \tag{2d}$$

Note that

$$\delta \int \frac{\mathrm{d}X^\lambda}{\mathrm{d}x^\lambda}\mathrm{d}^4 x = \delta \int_\sigma X^\lambda \mathrm{d}\sigma^\lambda = 0. \tag{2e}$$

Hence, only variations of *H* will contribute.

It is easy to see that variations of the function

$$H - 2k^\mu Q^\lambda_{\mu\nu}, \tag{2f}$$

where k^μ is a four-vector Lagrange multiplier, with respect to $\Gamma^\lambda_{(\mu\nu)}$, $Q^\lambda_{\mu\nu}$ and Γ_μ give, respectively, the three equations

$$s^{\mu\nu}_{,\lambda} + s^{\mu\alpha}\Gamma^\nu_{(\lambda\alpha)} + s^{\alpha\nu}\Gamma^\mu_{(\alpha\lambda)} - s^{\mu\nu}\Gamma^\alpha_{(\lambda\alpha)} = -[a^{\mu\alpha}Q^\nu_{\lambda\alpha} + a^{\alpha\nu}Q^\mu_{\alpha\lambda}], \tag{2g}$$

$$a^{\mu\nu}_{,\lambda} + a^{\mu\alpha}\Gamma^\nu_{(\lambda\alpha)} + a^{\alpha\nu}\Gamma^\mu_{(\alpha\lambda)} - a^{\mu\nu}\Gamma^\alpha_{(\lambda\alpha)} - k^\mu \delta^\nu_\lambda + k^\nu \delta^\mu_\lambda = -[s^{\mu\alpha}Q^\nu_{\lambda\alpha} + s^{\alpha\nu}Q^\mu_{\alpha\lambda}], \tag{2h}$$

and, $\ y a^{\mu\lambda}_{,\lambda} + x s^{\mu\nu}\Gamma_\nu = 0.$ $\tag{2i}$

It follows from the first two equations that

$$s^{\mu\alpha}_{,\alpha} + s^{\alpha\beta}\Gamma^\mu_{(\alpha\beta)} + a^{\alpha\beta}Q^\mu_{\alpha\beta} = 0, \tag{2j}$$

$$a^{\mu\nu}_{,\nu} = 3k^\mu. \tag{2k}$$

Hence, it follows from equation (2i) and the last equation that

$$k^\mu = \theta s^{\mu\nu}\Gamma_\nu, \qquad\qquad \theta = -\frac{x}{3y}\ , \tag{2l}$$

$$k^\mu_{,\mu} = 0. \tag{2m}$$

Equations (2k) and (2l) result in equation (27) in **#3**. Adding (2g) and (2h), we get

$$\bar{g}^{\mu\nu}_{,\lambda} + \bar{g}^{\mu\alpha}(\Gamma^\nu_{(\lambda\alpha)} + Q^\nu_{\lambda\alpha}) + \bar{g}^{\alpha\nu}(\Gamma^\mu_{(\alpha\lambda)} + Q^\mu_{\alpha\lambda}) - \bar{g}^{\mu\nu}\Gamma^\alpha_{(\lambda\alpha)} = k^\mu\delta^\nu_\lambda - k^\nu\delta^\mu_\lambda, \tag{2n}$$

Where, $\bar{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu}$. Multiplying (2n) by $\bar{g}_{\mu\nu}$ and using the results

$$\bar{g}^{\mu\nu}\ \bar{g}_{\mu\lambda} = \delta^\nu_\lambda, \qquad \bar{g}^{\mu\nu}\bar{g}_{\lambda\nu} = \delta^\mu_\lambda, \ Q^\lambda_{\alpha\lambda} = 0. \tag{2o}$$

We first observe that

$$\Gamma^{\alpha}_{(\lambda\alpha)} = \frac{|g|_{,\lambda}}{2\sqrt{-g}} + \frac{1}{2}(\bar{g}_{\lambda\beta} - \bar{g}_{\beta\lambda})k^{\beta}$$

$$\equiv \frac{|g|_{,\lambda}}{2\sqrt{-g}} + \bar{g}_{[\lambda\beta]}k^{\beta}. \tag{2p}$$

Hence, dividing (2n) by $\sqrt{-g}$, and also using (2p) and the results

$$g^{\mu\alpha}g_{\beta\alpha}k^{\beta} = k^{\mu} \qquad \text{and} \quad g^{\alpha\nu}g_{\alpha\beta}k^{\beta} = k^{\nu}, \tag{2q}$$

We get,

$$g^{\mu\nu}_{,\lambda} + g^{\mu\alpha}\left(\Gamma^{\nu}_{(\lambda\alpha)} + Q^{\nu}_{\lambda\alpha} + \frac{1}{\sqrt{-g}}(g_{\lambda\beta}k^{\beta}\delta^{\nu}_{\alpha} - g_{\beta\alpha}k^{\beta}\delta^{\nu}_{\lambda})\right) + g^{\alpha\nu}\left(\Gamma^{\mu}_{(\alpha\lambda)} + Q^{\mu}_{\alpha\lambda} + \frac{1}{\sqrt{-g}}(g_{\alpha\beta}k^{\beta}\delta^{\mu}_{\lambda} - g_{\beta\lambda}k^{\beta}\delta^{\mu}_{\alpha})\right)$$

$$= 3g^{\mu\nu}\frac{g_{[\lambda\beta]}k^{\beta}}{\sqrt{-g}}. \tag{2r}$$

Now, define the new affine coefficients

$$\Gamma''^{\nu}_{\lambda\alpha} = \left(\Gamma^{\nu}_{(\lambda\alpha)} + Q^{\nu}_{\lambda\alpha} + \frac{1}{\sqrt{-g}}(g_{\lambda\beta}k^{\beta}\delta^{\nu}_{\alpha} - g_{\beta\alpha}k^{\beta}\delta^{\nu}_{\lambda})\right), \tag{2s}$$

$$\Gamma''^{\mu}_{\alpha\lambda} = \left(\Gamma^{\mu}_{(\alpha\lambda)} + Q^{\mu}_{\alpha\lambda} + \frac{1}{\sqrt{-g}}(g_{\alpha\beta}k^{\beta}\delta^{\mu}_{\lambda} - g_{\beta\lambda}k^{\beta}\delta^{\mu}_{\alpha})\right) \tag{2t}$$

and

$$\Phi_{\lambda} = \frac{1}{2}g_{[\lambda\beta]}k^{\beta}, \tag{2u}$$

Which is equation (25) in **#3.** Then, equation (2r) can be written in the form

$$g^{\mu\nu}_{,\lambda} + g^{\mu\alpha}\Gamma''^{\nu}_{\lambda\alpha} + g^{\alpha\nu}\Gamma''^{\mu}_{\alpha\lambda} = 3g^{\mu\nu}\Phi_{\lambda}. \tag{2v}$$

This is equation (24) in **#3.**

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